Open and Closed Creations of Black Hole Pair

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Abstract

In the absence of a general no-boundary proposal for open creation, the constrained instanton approach is used in treating both the open and closed pair creations of black holes. A constrained instanton is considered as the seed for the quantum pair creation of black holes in the Kerr-Newman-(anti-)de Sitter family. At the WKB level, for the chargeless and nonrotating case, the creation probability is the exponential of the (minus) entropy of the universe. Also for the other cases (charged, rotating, or both), the creation probability is the exponential of (minus) one quarter of the sum of the inner and outer black hole horizon areas. The case of the Kerr-Newman family is also solved as a limiting case of that for the Kerr-Newman-anti-de Sitter family. The study of the open creation of a black hole pair can be considered as a prototype of the constrained instanton method to quantum gravity for an open universe, without appealing to the background subtraction approach.

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I. Introduction

In the No-Boundary Universe, the wave function of a closed universe is defined as a path integral over all compact 4-metrics with matter fields [1]. The dominant contribution to the path integral is from the stationary action solution. At the WKB level, the wave function can be written as

$$\Psi \approx e^{-I}, \tag{1}$$

where $I = I_r + iI_i$ is the complex action of the solution.

The Euclidean action is

$$I = -\frac{1}{16\pi}\int_M (R - 2\Lambda + L_m) - \frac{1}{8\pi}\oint_{\partial M} K, \tag{2}$$

where R is the scalar curvature of the spacetime M, K is the trace of the second form of the boundary ∂M, Λ is the cosmological constant, and L_m is the Lagrangian of the matter content.

The imaginary part I_i and real part I_r of the action represent the Lorentzian and Euclidean evolutions in real time and imaginary time, respectively. When their orbits are intertwined they are mutually perpendicular in the configuration space with the supermetric. The probability of a Lorentzian orbit remains constant during the evolution. One can identify the probability, not only as the probability of the universe created, but also as the probabilities for other Lorentzian universes obtained through an analytic continuation from it [2].

An instanton is defined as a stationary action orbit and satisfies the Einstein equation everywhere, and is the seed for the creation of the universe. However, very few regular instantons exist. The framework of the No-Boundary Universe is much wider than that of the instanton theory. Therefore, in order not to exclude many interesting phenomena from the study, one has to appeal to the concept of constrained instantons [3]. Constrained instantons are the orbits with an action which is stationary under some restriction. The restriction can be imposed on a spacelike 3-surface of the created Lorentzian universe. This restriction is that the 3-metric and matter content are given at the 3-surface. The relative creation probability from the instanton is the exponential of the negative of the real part of the instanton action.

The usual prescription for finding a constrained instanton is to obtain a complex solution to the Einstein equation and other field equations in the complex domain of spacetime coordinates. If there is no singularity in a compact section of the solution, then the section is considered as an instanton. If there exist singularities in the section, then the action of the section is not stationary. The action may only be stationary with respect to the variations under some restrictions mentioned above. If this is the case, then the section is a constrained gravitational instanton. To find the constrained instanton, one has to closely investigate the singularities. The stationary action condition is crucial to the validation of the WKB approximation. We are going to work at the WKB level for the problem of quantum creation of a black hole pair.

A main unresolved problem in quantum cosmology is to generalize the no-boundary proposal for an open universe. While a general prescription is not available, one can still use analytic continuation to obtain the WKB approximation to the wave function for open universes with some kind of symmetry.

The most symmetric space is the S^4 space with $O(5)$ symmetry, or the four-sphere,

$$ds^2 = d\tau^2 + \frac{3}{\Lambda}\cos^2\left(\sqrt{\frac{\Lambda}{3}}\tau\right)(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)), \tag{3}$$

where Λ is a positive cosmological constant. One can obtain the de Sitter space with 3-spheres as the spatial sections of constant t by the substitution $\tau = it$. The de Sitter space with 3-hyperboloids as the spatial sections of constant t is obtained by the substitutions $\tau = \sqrt{\frac{3}{\Lambda}}\frac{\pi}{2} + it$ and $\chi = i\rho$.

One can also obtain the anti-de Sitter space by the substitution $\chi = i\rho$. The signature of anti-de Sitter space is $(+,-,-,-)$. This signature associated with the anti-de Sitter space is reasonable, since the relative sign of the cosmological constant is implicitly changed by the analytic continuation. If one prefers the usual signature of the anti-de Sitter space, then he could start from the four-sphere with the signature of $(-,-,-,-)$, instead of (3) [4].

One can reduce the $O(5)$ symmetry to make the model more realistic. This is the $FLRW$ space with $O(4)$ symmetry,

$$ds^2 = d\tau^2 + a^2(\tau)(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)), \tag{4}$$

where $a(\tau)$ is the length scale of the homogeneous 3-surfaces and $a(0) = 0$. One can apply the combined analytic continuations $t = -i\tau$ and $\chi = i\rho$ to obtain an open $FLRW$ universe [2].

The study of the perturbation modes around this background, strictly following the no-boundary philosophy, is waiting for a general proposal for the quantum state of an open universe. If one includes "realistic" matter fields in the model, then the instanton is not regular. However, the singular instanton can be interpreted as a constrained instanton [5].

In this paper, we try to reduce the symmetry further, that is to investigate vacuum models with $O(3)$, or spherical symmetry. This will lead to a quantum pair creations of black holes in the de Sitter (anti-de Sitter) space background. We shall consider the vacuum model with a positive cosmological constant first. There exist only two regular instantons, they are the S^4 and $S^2 \times S^2$ spaces. These are the origins of the de Sitter space and the Nariai space [6]. The Nariai space is interpreted as a pair of black holes with maximal mass $m_c = \Lambda^{-1/2}/3$. For the case of sub-maximal black hole pair creation, one has to use a real constrained gravitational instanton as the seed for the creation. If the cosmological constant is negative, one has to use a complex constrained instanton as the seed for a pair creation of Schwarzschild-anti-de Sitter black holes.

This argument is generalized to the pair creations of all other members of the Kerr-Newman black hole family. For the pair creation of black holes in the de Sitter space background, the real constrained instantons becomes the seeds, while for the pair creation of black holes in the anti-de Sitter space background, one has to use the complex constrained instantons as the seeds. We shall investigate the pair creation of the Schwarzschild, Reissner-Nordström, Kerr and Newman black holes in Sects. II, III, IV and V, respectively. For the rotating black hole case, the symmetry has been reduced to $O(2)$. All these instantons have an extra $U(1)$ isometry that agrees with the time invariance.

The pair creation probability of Schwarzschild black holes in the closed de Sitter space background is the exponential of the entropy of the universe, while in the open anti-de Sitter space background the probability is the exponential of the negative of the entropy. This is reasonable, since it implies that the more massive the black holes are, the more unlikely they are to be created.

For the pair creations of other members in the Kerr-Newman family in the closed (open) backgrounds, the creation probability is the exponential of (minus) one quarter of the sum of the outer and inner black hole horizon areas. The de Sitter (anti-de Sitter) spacetime without a black hole is the most probable evolution comparing with that with a pair of black holes.

Sect. VI will discuss gravitational thermodynamics using constrained instantons. The independence of the action from the identification time period implies that the action is equal to the negative of the entropy. However, one has to clarify the physical meaning of the entropy. Sect.VII is a discussion. It is argued that to respect the principle of general covariance, one has to work in the domain of complex coordinates in quantum gravity. This justifies the quantum creation of a black hole pair from our complex constrained instanton.

II. Schwarzschild

The solution to the Einstein equation is written

$$ds^2 = \Delta d\tau^2 + \Delta^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5}$$

$$\Delta = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}, \tag{6}$$

where m is an integral constant. One can make a factorization

$$\Delta = -\frac{\Lambda}{3r}(r - r_0)(r - r_2)(r - r_3). \tag{7}$$

One can define

$$\alpha = \frac{1}{3}\arccos(3m\Lambda^{1/2}), \tag{8}$$

then one has

$$r_2 = 2\sqrt{\frac{1}{\Lambda}}\cos\left(\alpha + \frac{\pi}{3}\right), \quad r_3 = 2\sqrt{\frac{1}{\Lambda}}\cos\left(\alpha - \frac{\pi}{3}\right), \tag{9}$$

$$r_0 = -2\sqrt{\frac{1}{\Lambda}}\cos\alpha. \tag{10}$$

The surface gravity κ_i of r_i is [7]

$$\kappa_i = \frac{\Lambda}{6r_i}\prod_{j=0,2,3,\ (j\neq i)}(r_i - r_j). \tag{11}$$

If Λ is positive and $0 \leq m \leq m_c$, then r_2 and r_3 are real. There exist regular instantons S^4 and $S^2 \times S^2$ for the cases $m = 0$ and $m = m_c$, respectively. The case $m = 0$ leads to the creation of a universe without a black hole and the case $m = m_c$ leads to the creation of a universe with a pair of

maximal black holes [6]. For the general case, one can make a constrained instanton as follows. The constrained instanton is the seed for the quantum creation of a Schwarzschild-de Sitter black hole pair, or a sub-maximal black hole pair [3], and r_2 and r_3 become the black hole and cosmological horizons for the holes created.

One can have two cuts at $\tau = \pm\Delta\tau/2$ between the two horizons. Then the f_2-fold cover around the black hole horizon $r = r_2$ turns the $(\tau - r)$ plane into a cone with a deficit angle $2\pi(1 - f_2)$ there. In a similar way one can have an f_3-fold cover at the cosmological horizon. In order to form a fairly symmetric Euclidean manifold, one can glue these two cuts under the condition

$$f_2\beta_2 + f_3\beta_3 = 0, \tag{12}$$

where $\beta_i = 2\pi\kappa_i^{-1}$ are the periods of τ that avoid conical singularities in compacting the Euclidean spacetime at these two horizons, respectively. The absolute values of their reciprocals are the Hawking temperature and the Gibbons-Hawking temperature. If f_2 or f_3 is different from 1 (at least one should be, since the two periods are different for the sub-maximal black holes), then the cone at the black hole or cosmological horizon will have an extra contribution to the action of the instanton. After the transition to Lorentzian spacetime, the conical singularities will only affect the real part of the phase of the wave function, i.e. the probability of the black hole creation.

The extra contribution due to the conical singularities can be considered as the degenerate form of the surface term in the action (2) and can be written as follows:

$$I_{i,deficit} = -\frac{1}{8\pi} \cdot 4\pi r_i^2 \cdot 2\pi(1 - f_i). \quad (i = 2, 3) \tag{13}$$

The volume term of the action for the instanton can be calculated

$$I_{vol} = -\frac{\Lambda}{6}(r_3^3 - r_2^3) f_2\beta_2. \tag{14}$$

Using eqs. (11) - (14), one obtains the total action

$$I = -\pi(r_2^2 + r_3^2). \tag{15}$$

This is one quarter of the negative of the sum of the two horizon areas. One quarter of the sum is the total entropy of the universe.

It is remarkable to note that the action is independent of the choice of f_2 or f_3. Our manifold satisfies the Einstein equation everywhere except for the two horizons at the equator. The equator is two joint sections $\tau = consts.$ passing these horizons. It divides the instanton into two halves. The Lorentzian metric of the black hole pair created can be obtained through an analytic continuation of the time coordinate from an imaginary to real value at the equator. We can impose the restriction that the 3-geometry characterized by the parameter m is given at the equator, i.e. the transition surface. The parameter f_2 or f_3 is the only degree of freedom left, since the field equation holds elsewhere. Thus, in order to check whether we get a stationary action solution for the given horizons, one only needs to see whether the above action is stationary with respect to this parameter. Our result (15) shows that our gravitational action has a stationary action and the manifold is qualified as a constrained instanton. The exponential of the negative of the action can be used for the WKB approximation to the probability.

Eq. (15) also implies that no matter which value of f_2 or f_3 is chosen, the same black hole should be created with the same probability. Of course, the most dramatic case is the creation of a universe from no volume, i.e. $f_2 = f_3 = 0$.

From eq. (15) it follows that the relative probability of the pair creation of black holes in the de Sitter background is the exponential of the total entropy of the universe [3] [8].

One can study quantum no-boundary states of scalar and spinor fields in this model. It turns out that these fields are in thermal equilibrium with the background. The associated temperature is the reciprocal of the identification time period as expected, and it can take an arbitrary value [9].

Now, let us discuss the case of $\Lambda < 0$. One is interested in the probability of pair creation of Schwarzschild-anti-de Sitter black holes. The universe is open. Hence, our key point is to find a complex solution which has both the universe as its Lorentzian section and a compact section as the seed for the creation, i.e. the constrained instanton. The real part of its action will determine the creation probability.

The metric of the constrained instanton takes the same form as eq. (5). However, two zeros of Δ become complex conjugates. One can define

$$\gamma \equiv \frac{1}{3}\mathrm{arcsinh}(3m|\Lambda|^{1/2})), \tag{16}$$

and then one has

$$r_2 = 2\sqrt{\frac{1}{|\Lambda|}}\sinh\gamma,$$

$$r_3 = \bar{r}_0 = \sqrt{\frac{1}{|\Lambda|}}(-\sinh\gamma - i\sqrt{3}\cosh\gamma). \tag{17}$$

One can build a complex constrained instanton using the section connecting r_0 and r_3. Since r_0 and r_3 are complex conjugates, the real part of r on the section is constant, and the range of the imaginary part runs between $\pm i\sqrt{\frac{3}{|\Lambda|}}\cosh\gamma$. The surface gravities κ_0 and κ_3 are complex conjugates too. Following the procedure of constructing the constrained gravitational instanton for the case $\Lambda > 0$, we can use complex folding parameters f_0 and f_3 to cut, fold and glue the complex manifold with

$$f_0\beta_0 + f_3\beta_3 = 0. \tag{18}$$

As expected, the action is independent of the parameter f_0 or f_3 and

$$I = -\pi(r_0^2 + r_3^2) = \pi\left(-\frac{6}{\Lambda} + r_2^2\right). \tag{19}$$

The action is independent from the choice of the time identification period. One can always choose the arbitrary time identification period to be imaginary, and the Lorentzian section in which we are living is associated with the real time. A special choice of the imaginary time period will regularize the conical singularity of the Euclidean section at the black hole horizon r_2. However, we do not have to do so, since constrained instantons are allowed in quantum cosmology.

One can obtain the Lorentzian metric from an analytic continuation of the time coordinate from an imaginary to real value at the equator of the instanton. The equator is two joint $\tau = consts.$ sections passing through these horizons. The 3-geometry of the equator can be considered as the restriction imposed for the constrained instanton. Again, the independence of (19) from the time identification period shows that the manifold is qualified as a constrained instanton.

Therefore, the relative probability of the pair creation of Schwarzschild-anti-de Sitter black holes, at the WKB level, is the exponential of the negative of one quarter of the black hole horizon area, in contrast to the case of pair creation of black holes in the de Sitter space background. One quarter of the black hole horizon area is known to be the entropy in the Schwarzschild-anti-de Sitter universe [10].

One may wonder why we choose horizons r_0 and r_3 to construct the instanton. One can also consider those constructions involving r_2 as the instantons. However, the real part of the action for our choice is always greater than that of the other choices for the given configuration, and the wave function or the probability is determined by the classical orbit with the greatest real part of the action [1]. When we dealt with the Schwarzschild-de Sitter case, the choice of the instanton with r_2 and r_3 had the greatest action accidentally, but we did not appreciate this earlier.

III. Reissner-Nordström

Now, let us include the Maxwell field into the model. The Reissner-Nordström-de Sitter spacetime, with mass parameter m, charge Q and a positive cosmological constant Λ, is the only spherically symmetric electrovac solution to the Einstein and Maxwell equations. Its Euclidean metric can be written as

$$ds^2 = \Delta d\tau^2 + \Delta^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{20}$$

where Δ is

$$\Delta = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}\right). \tag{21}$$

For convenience one can make a factorization

$$\Delta \equiv -\frac{\Lambda}{3r^2}(r - r_0)(r - r_1)(r - r_2)(r - r_3), \tag{22}$$

where r_0, r_1, r_2, r_3 are roots of Δ in ascending order.

The gauge field is

$$F = -\frac{iQ}{r^2} d\tau \wedge dr \tag{23}$$

for an electrically charged solution, and

$$F = Q \sin\theta d\theta \wedge d\phi \tag{24}$$

for a magnetically charged solution. We shall not consider dyonic solutions.

The roots satisfy the following relations

$$\sum_i r_i = 0, \tag{25}$$

$$\sum_{i>j} r_i r_j = -\frac{3}{\Lambda}, \tag{26}$$

$$\sum_{i>j>k} r_i r_j r_k = -\frac{6m}{\Lambda} \tag{27}$$

and

$$\prod_i r_i = -\frac{3Q^2}{\Lambda}. \tag{28}$$

If all the roots r_0, r_1, r_2, r_3 are real, then r_0 is negative, r_2 and r_3 are identified as the inner and outer black hole horizons, and r_3 is the cosmological horizon.

The surface gravity κ_i of r_i is [7]

$$\kappa_i = \frac{\Lambda}{6r_i^2} \prod_{j=0,1,2,3,\ (j\neq i)} (r_i - r_j). \tag{29}$$

One can make a real constrained gravitational instanton by gluing two sections of constant values of imaginary time τ between the two complex horizons r_1 and r_2. Then the f_i-fold $(i = 1, 2)$ cover turns the $(\tau - r)$ plane into a cone with a deficit angle $2\pi(1 - f_i)$ at the horizons. Both f_1 and f_2 can take any pair of complex numbers with the relation

$$f_1\beta_1 + f_2\beta_2 = 0, \tag{30}$$

where $\beta_i = 2\pi\kappa_i^{-1}$. If f_1 or f_2 is different from 1, then the cone at the complex horizon will have an extra contribution to the action of the instanton. One can choose f_1 to make the time identification period to be imaginary. The Lorentzian metric for the black hole pair created can be obtained by analytic continuation of the imaginary time to real time at the equator of the constrained instanton. The equator is two joint sections $\tau = consts.$ passing the two horizons. It divides the instanton into two halves, and has topology $S^2 \times S^1$. After the transition to Lorentzian spacetime, the conical singularities will affect the real part of the phase of the wave function, i.e. the probability of the creation of the black holes.

The contributions to the action due to the conical singularities at the horizons are

$$I_{i,deficit} = -\frac{1}{8\pi} \cdot 4\pi r_i^2 \cdot 2\pi(1 - f_i). \ \ (i = 1, 2) \tag{31}$$

They are degenerate forms of the surface terms.

The action due to the volume is

$$I_v = -\frac{f_1\beta_1\Lambda}{6}(r_2^3 - r_1^3) \pm \frac{f_1\beta_1 Q^2}{2}(r_1^{-1} - r_2^{-1}), \tag{32}$$

where $+$ is for the magnetic case and $-$ is for the electric case.

In the magnetic case, the boundary date is h_{ij} and A_i. The vector potential, in turn, determines the magnetic charge, since it can be obtained by the magnetic flux, or the integral of the gauge field F over the S^2 factor. It is more convenient to choose a gauge potential

$$A = Q(1 - \cos\theta)d\phi \tag{33}$$

to evaluate the flux.

In the electric case, the boundary data is h_{ij} and the momentum ω [3][11][12], which is canonically conjugate to the electric charge and defined by

$$\omega = \int A, \tag{34}$$

where the integral is around the S^1 direction. The most convenient choice of the gauge potential for the calculation is

$$A = -\frac{iQ}{r^2}\tau dr. \tag{35}$$

The wave function for the equator is the exponential of half the negative of the action. For the magnetic cases, one obtains the wave function $\Psi(h_{ij})$ and $\Psi(Q, h_{ij})$. For the electric case, what one obtains this way is $\Psi(\omega, h_{ij})$ instead of $\Psi(Q, h_{ij})$. One can get the wave function $\Psi(Q, h_{ij})$ for a given electric charge through the Fourier transformation

$$\Psi(Q, h_{ij}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega Q}\Psi(\omega, h_{ij}). \tag{36}$$

This Fourier transformation is equivalent to a multiplication of an extra factor

$$\exp\left(\frac{-f_1\beta_1 Q^2(r_1^{-1} - r_2^{-1})}{2}\right) \tag{37}$$

to the wave function. This makes the probabilities for magnetic and electric cases equal, and thus recovers the duality between the magnetic and electric black holes [3][11][12].

Therefore, the effective action for both the magnetic and electric case is

$$I = -\pi(r_1^2 + r_2^2). \tag{38}$$

The effective action is independent from the choice of the time identification period. By the same argument, the manifold constructed is qualified as a constrained instanton.

The relative probability of the universe creation is the exponential of the negative of the seed instanton action. In our case, this is the exponential of the sum of the outer and inner black hole horizon areas.

It is noted that if one has not used the Fourier transformation (36), then one cannot obtain the constrained instanton and the whole calculation becomes meaningless.

The reason that we use the inner and outer black hole horizons to construct the constrained instanton is that the instanton has the largest action in comparison with other options, as can be simply proven. For the same configuration the orbit with the largest real part of the action determines the wave function and probability [1].

Now let us discuss the pair creation in the anti-de Sitter space background. The spherically symmetric electrovac model with a negative cosmological constant is also described by eqs. (20)-(24). For the general case, one has two real horizons r_2, r_3, which are identified as the inner and outer black hole horizons. The other two horizons r_0, r_1 are complex conjugates. One then uses these two complex horizons to construct the complex constrained instanton as in the Schwarzschild-anti-de Sitter case. The action is

$$I = -\pi(r_0^2 + r_1^2) = \pi\left(-\frac{6}{\Lambda} + r_2^2 + r_3^2\right), \tag{39}$$

where we use the fact that the sum of all horizon areas is equal to $24\pi\Lambda^{-1}$, which is implied by eqs. (25)(26).

The relative creation probability is the exponential of the negative of the seed instanton action. In our case, this becomes the exponential of the negative of the sum of the outer and inner black hole horizon areas.

This choice of the horizons for the construction is again justified by the fact that, the action of the instanton considered is largest for the configuration of the wave function [1]. This point is important. For example, if, instead we use r_2 and r_3 for constructing the instanton, then the creation

probability of a universe without a black hole would be smaller than that with a pair of black holes. This is physically absurd.

The pair creation of Reissner-Nordström-de Sitter black holes from regular instantons was studied in [11][12][13][14]. These become the special cases of the discussion presented in [3]. The pair creation of compactified Reissner-Nordström-anti-de Sitter black holes was discussed in [15][16]. However, to avoid the difficulty associated with open creation, a domain wall was introduced to compactify the non-compact geometry.

IV. Kerr

Now let us discuss the creation of a rotating black hole in the (anti-)de Sitter space background. The Lorentzian metric of the black hole spacetime is [7]

$$ds^2 = \rho^2(\Delta_r^{-1}dr^2 + \Delta_\theta^{-1}d\theta^2) + \rho^{-2}\Xi^{-2}\Delta_\theta \sin^2\theta(adt - (r^2+a^2)d\phi)^2 - \rho^{-2}\Xi^{-2}\Delta_r(dt - a\sin^2\theta d\phi)^2, \tag{40}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{41}$$

$$\Delta_r = (r^2+a^2)(1-\Lambda r^2 3^{-1}) - 2mr + Q^2 + P^2, \tag{42}$$

$$\Delta_\theta = 1 + \Lambda a^2 3^{-1}\cos^2\theta, \tag{43}$$

$$\Xi = 1 + \Lambda a^2 3^{-1} \tag{44}$$

and m, a, Q and P are constants, m and ma representing mass and angular momentum. Q and P are electric and magnetic charges.

One can factorize Δ_r as follows

$$\Delta_r = -\frac{\Lambda}{3}(r-r_0)(r-r_1)(r-r_2)(r-r_3). \tag{45}$$

The roots r_i satisfy the following relations:

$$\sum_i r_i = 0, \tag{46}$$

$$\sum_{i>j} r_i r_j = -\frac{3}{\Lambda} + a^2, \tag{47}$$

$$\sum_{i>j>k} r_i r_j r_k = -\frac{6m}{\Lambda}, \tag{48}$$

$$\prod_i r_i = -\frac{3(a^2 + Q^2 + P^2)}{\Lambda}. \tag{49}$$

We shall concentrate on the neutral case with $Q = P = 0$ first. The charged case with nonzero electric or magnetic charge will be discussed later.

For the general closed case with a positive cosmological constant, one root, say r_0, is negative, one can identify the other three positive roots r_1, r_2, r_3 as the inner black hole, outer black hole and cosmological horizons, respectively.

The probability of the Kerr-de Sitter black hole pair creation, at the WKB level, is the exponential of the negative of the action of its constrained gravitational instanton. In order to form a constrained gravitational instanton, one can do the similar cutting, folding and covering between the inner and outer black hole horizons with folding parameters f_1 and f_2 satisfying relation (30) as in the nonrotating case. The reason to consider the construction with horizons r_1 and r_2 as the seed instanton is the same: It has the largest action for the same configuration of the wave function comparing with other choices. The Lorentzian metric for the black hole pair created is obtained through analytic continuation in the same way as for the nonrotating case. The equator where the quantum transition will occur has topology $S^2 \times S^1$. The restrictions can be similarly imposed at the equator for the constrained instanton.

The horizon areas are

$$A_i = 4\pi (r_i^2 + a^2)\Xi^{-1}. \tag{50}$$

The surface gravities of the horizons are

$$\kappa_i = \frac{\Lambda \prod_{j\ (j \neq i)} (r_i - r_j)}{6\Xi (r_i^2 + a^2)}. \tag{51}$$

The actions due to the horizons are

$$I_{i,horizon} = -\frac{\pi (r_i^2 + a^2)(1 - f_i)}{\Xi}. \ \ (i = 1, 2) \tag{52}$$

The action due to the volume is

$$I_v = -\frac{f_1\beta_1\Lambda}{6\Xi^2}(r_2^3 - r_1^3 + a^2(r_2 - r_1)), \tag{53}$$

where we define $\beta_i = 2\pi\kappa_i^{-1}$.

If one naively takes the exponential of the negative of half the total action, then the exponential is not identified as the wave function at the creation moment of the black hole pair. The physical reason is that what one can observe is only the angular differentiation, or the relative rotation of the two horizons. This situation is similar to the case of a Kerr black hole pair in the asymptotically flat background. There one can only measure the rotation of the black hole horizon from the spatial infinity. To find the wave function for the given mass and angular momentum one has to make the Fourier transformation [3]

$$\Psi(m, a, h_{ij}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\delta e^{i\delta J\Xi^{-2}}\Psi(m, \delta, h_{ij}), \tag{54}$$

where δ is the relative rotation angle for the half time period $f_1\beta_1/2$, which is canonically conjugate to the angular momentum $J = ma$; and the factor Ξ^{-2} is due to the time rescaling. The angle difference δ can be evaluated

$$\delta = \int_0^{f_1\beta_1/2} d\tau(\Omega_1 - \Omega_2), \tag{55}$$

where the angular velocities at the horizons are

$$\Omega_i = \frac{a}{r_i^2 + a^2}. \tag{56}$$

The Fourier transformation is equivalent to adding an extra term into the action for the constrained instanton, and then the total action becomes

$$I = -\pi(r_1^2 + a^2)\Xi^{-1} - \pi(r_2^2 + a^2)\Xi^{-1}. \tag{57}$$

It is crucial to note that the action is independent of the identification time period β, $f_1\beta_1$ for our case, and therefore, the manifold obtained is qualified as a constrained instanton. Again, if one does not adapt the Fourier transformation between the angular momentum and the relative rotation angle, then one cannot obtain the constrained instanton.

Therefore, the relative probability of the Kerr black hole pair creation is

$$P_k \approx \exp(\pi(r_1^2 + a^2)\Xi^{-1} + \pi(r_2^2 + a^2)\Xi^{-1}). \tag{58}$$

It is the exponential of one quarter of the sum of the outer and inner black hole horizon areas.

For the general open case with a negative cosmological constant, there exist at least two complex conjugate roots, say r_0, r_1. We assume the other two roots r_2, r_3 to be real and identify them as the inner and outer black hole horizons. By the same reason, in order to obtain an instanton with the greatest action for a given configuration, one has to choose complex horizons r_0, r_1 to make a constrained instanton.

After the Fourier transformation associated with the rotation, one can show that the construction is indeed the instanton required. It is the seed for the pair creation of Kerr-anti-de Sitter black holes. The action is

$$I = -\pi(r_0^2 + a^2)\Xi^{-1} - \pi(r_1^2 + a^2)\Xi^{-1} = \pi\left(-\frac{6}{\Lambda} + (r_2^2 + a^2)\Xi^{-1} + (r_3^2 + a^2)\Xi^{-1}\right), \qquad (59)$$

where we use the fact that the sum of all horizon areas is $24\pi\Lambda^{-1}$, which can be derived from eqs. (46)(47).

Therefore, the relative probability is

$$P_k \approx \exp -(\pi(r_2^2 + a^2)\Xi^{-1} + \pi(r_3^2 + a^2)\Xi^{-1}). \qquad (60)$$

It is the exponential of the negative of one quarter of the sum of the inner and outer black hole horizon areas.

V. Newman

Now, let us turn to the charged and rotating black hole case. The vector potential can be written as

$$A = \frac{Qr(dt - a\sin^2\theta d\phi) + P\cos\theta(adt - (r^2 + a^2)d\phi)}{\rho^2}. \qquad (61)$$

We shall not consider the dyonic case in the following.

One can closely follow the neutral rotating case for calculating the action of the corresponding constrained gravitational instanton. The only difference is to add one more term due to the electromagnetic field to the action of volume. For the magnetic case, it is

$$\frac{f_1\beta_1 P^2}{2\Xi^2}\left(\frac{r_i}{r_i^2 + a^2} - \frac{r_j}{r_j^2 + a^2}\right) \qquad (62)$$

and for the electric case, it is

$$-\frac{f_1\beta_1 Q^2}{2\Xi^2}\left(\frac{r_i}{r_i^2+a^2}-\frac{r_j}{r_j^2+a^2}\right), \tag{63}$$

where (i,j) is $(1,2)$ and $(0,1)$ for the closed and open cases, respectively.

In the magnetic case the vector potential determines the magnetic charge, which is the integral over the S^2 factor. Putting all these contributions together one can find

$$I = -\pi(r_i^2+a^2)\Xi^{-1} - \pi(r_j^2+a^2)\Xi^{-1}, \tag{64}$$

and the relative probabilities of the closed and open pair creations of magnetically charged black holes is written in the same form as eqs. (58)(60).

In the electric case, one can only fix the integral

$$\omega = \int A, \tag{65}$$

where the integral is around the S^1 direction.

So, what one obtains in this way is $\Psi(\omega, a, h_{ij})$. In order to get the wave function $\Psi(Q, a, h_{ij})$ for a given electric charge, we have to repeat the procedure like the Reissner-Nordström case. The Fourier transformation is equivalent to adding one more term to the action

$$\frac{f_1\beta_1 Q^2}{\Xi^2}\left(\frac{r_i}{r_i^2+a^2}-\frac{r_j}{r_j^2+a^2}\right). \tag{66}$$

Then we obtain the same probability formulas for the electrically charged rotating black hole pair creation as for the magnetic one. The duality between the magnetic and electric cases is recovered.

The pair creation of Kerr-Newman-de Sitter black holes from a regular instanton was also studied [17], as a special case of the general discussion [3].

VI. Thermodynamics

In the constrained instanton approach to quantum cosmology associated with black hole pair creation, the fact that the action is independent from the imaginary time period β is crucial.

In gravitational thermodynamics, the partition function Z is identified with the path integral over all metrics g and matter fields ψ on a manifold [18],

$$Z = \int d[g]d[\psi] \exp -I(g, \psi). \tag{67}$$

The WKB approximation to the path integral is equivalent to the contribution of the background, excluding the fluctuations. The background is the stationary action orbit, or the constrained instanton. At this level, one has

$$Z = \exp -I. \tag{68}$$

For the regular compact instanton case, there are no externally imposed quantities and corresponding chemical potentials. Therefore, the partition function simply counts the total number of the states, and each state is equally probable with the probability $p_n = Z^{-1}$. Thus, the entropy is $S = -p_n \log p_n = \log Z$. Therefore, at the WKB level, the entropy is the negative of the action of the instanton [19].

For the regular noncompact instanton case, like the Kerr-Newman-anti-de Sitter family, the system is constrained by three quantities, namely mass or energy m, electric charge Q and angular momentum J. There are two corresponding chemical potentials, the electrostatic potential Φ at the outer black hole horizon and angular velocity Ω. Then one has to use the grand partition function Z in grand canonical ensembles for the thermodynamics study [18],

$$Z = \mathrm{Tr} \exp(-\beta m + \beta \Omega J + \beta \Phi Q) = \int d[g]d[\psi] \exp -I(g, \psi). \tag{69}$$

Here, the path integral is over all fields whose value at the point $(\tau - \beta, r, \theta, \phi + i\beta\Omega)$ is $\exp(Q\beta, \phi)$ times the value at (τ, r, θ, ϕ) [20].

Strictly speaking, the quantity β of the term βm is the difference of the Euclidean time lapse at infinity, which is β, and that at the outer black hole horizon, which is zero. The quantity $\beta\Omega J$ is the difference of the rotation angles measured at the horizon and at infinity during the Euclidean time lapses. The quantity $\beta\Phi$ is the difference of the electrostatic potentials at the horizon and at infinity.

For the constrained instanton case, the imposed quantities are given by the constraints for the instanton. The partition function (69) remains valid, but the relevant quantities should be interpreted

as the differences between that at the two horizons used for constructing the instanton, instead of that at the outer black hole horizon and at infinity. Since the Euclidean time lapses at two horizons are zero, the term βm disappears in the exponent of (69).

The dominant contribution to the path integral (69) is due to a stationary action orbit. However, there does not exists such an orbit satisfying the jump condition for nonzero Q or J. It is noted that a is real. Instead, the WKB approximation of the path integral is the exponential of the negative of the effective action of the constrained instanton constructed earlier. The instanton is obtained from the cutting, folding, gluing of the complex solution. If needed, the two Fourier transformations are introduced. This is an alternative justification of the two Fourier transformations.

Even for the compact instanton case, if the instanton is electrically charged, or rotating, or both, then the naively evaluated partition function without imposed quantities as in (67), does not correspond to what we want. One has to impose the quantities Q, or J, or both, and use (69) instead.

The entropy S can be obtained by

$$S = -\frac{\beta\partial}{\partial\beta}\ln Z + \ln Z = -I, \tag{70}$$

where the action I is the effective one, of course.

Thus, the condition that I is independent from β implies that the "entropy" is the negative of the action.

For the closed creation of the chargeless and nonrotating black holes, the "entropy" is the true entropy of the universe. For the closed creation of the charged or (and) rotating black holes, the "entropy" is one quarter of the sum of the inner and outer black hole horizon areas. For compact regular instantons, the fact that the entropy is equal to the negative of the action is shown using different arguments in [19].

It is noted from eqs. (25)(26)(46) and (47) that, for all members of the Kerr-Newman-(anti-)de Sitter family, the sum of all horizon areas is equal to $24\pi\Lambda^{-1}$. This fact seems coincidental, but it has a deep physical significance.

For the open creation of the chargeless and nonrotating black holes, the "entropy" is associated with the complex horizons. It becomes the negative of the true entropy of the universe, up to a constant $6\pi\Lambda^{-1}$. Equivalently, the action is the entropy up to the same constant. This constant

is ignored in the background subtraction approach anyway. For the open creation of the charged or (and) rotating black holes, the "entropy" becomes one quarter of the negative of the sum of the inner and outer black hole horizon areas up to the constant.

Using a standard technique designed for spaces with spatially noncompact geometries [18], the action of the Kerr-Newman-anti-de Sitter space is evaluated as follows: The physical action is defined by the difference between the action of the space under study and that of a reference background. The background can be a static solution to the field equation. From gravitational thermodynamics, one can derive the entropy from the action.

In quantum gravity the quantum state can be represented by a matrix density. Apparently, the state associated with our constrained instanton is an eigenstate of the entropy operator, instead of the temperature operator, as previously thought.

VII. Discussion

The Hawking temperature is defined as the reciprocal of the absolute value of the time identification period required to make the Euclidean manifold regular at the horizon. In the background subtraction approach for an open universe, if one lifts the regularity condition at the horizon, or lets the period take an arbitrary value, then one finds that the action does depend on the period and becomes meaningless. However, if we calculate the action using our complex constrained instanton, then the action is independent of the complex period β. It is noted that the values of the action are different for these two methods. The beautiful aspect of our approach is that, even in the absence of a general no-boundary proposal for open universes, we can treat the creation of the closed and the open universes in the same way.

Our treatment of quantum creation of the Kerr-Newman-anti-de Sitter space using the constrained instanton can be thought of as a prototype of quantum gravity for an open system without appealing to the background subtraction approach.

The Kerr-Newman black hole case can be thought of as the limit of our case as we let Λ approach 0 from below. The constrained instanton approach naturally explains the fact that the action and entropy of a Schwarzschild space are equal.

In the constrained instanton approach, we use the effective action after introducing the two Fourier transformations. This is equivalent to the requirements for the proper evaluation of (69) in grand canonical ensembles. Since the effective action is independent of β, then the "entropy" S becomes the negative of the effective action I. However, one has to clarify the meaning of the "entropy."

When we construct the constrained instanton, we always select two horizons such that the action takes its largest value. Only for the case of Schwarzschild-de Sitter black hole pair creation, does the quantum transition occur at a true spacelike 3-surface. For the rest of the cases, the transitions seem quite counter-intuitive. People have spent a lot of effort developing quantum gravity for spacetimes with a $U(1)$ isometry. The Killing time associated with the isometry is imaginarized to obtain the Euclidean orbits. This procedure is justified by the isometry. However, for general spacetimes, there does not exist a prestigious time coordinate. From the principle of general covariance, one has to work in the domain of all complex coordinates. This is the reason that one has to live with the quantum transition through any kind of 3-surface.

In summary, for the chargeless and nonrotating case, the closed (open) creation probability is the exponential of the (minus) entropy of the universe, and for the other cases (charged, rotating, or both), the creation probability is the exponential of (minus) one quarter of the sum of the inner and outer black hole horizon areas.

Due to the black hole No-Hair Theorem, the problem of black hole pair creation in both the de Sitter and the anti-de Sitter space backgrounds has been completely resolved.

It can be shown that the probability of the universe creation without a black hole is greater than that with a pair of black holes in both the de Sitter and the anti-de Sitter backgrounds.

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